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Hamiltonian Reduction and Supersymmetric Toda Models

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Abstract

New formulations of the solutions of N=1 and N=2 super Toda field theory are introduced, using Hamiltonian reduction of the N=1 and N=2 super WZNW Models to the super Toda Models. These parameterisations are then used to present the Hamiltonian formulations of the super Toda theories on the spaces of solutions.

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1 Introduction

Super Toda theories [1, 2, 3, 4, 5] are two dimensional conformal field theories and are an arena for the investigation of general features of two dimensional integrable systems such as quantum supergroups, super W-algebras, and supersymmetric integrable hierarchies. The study of many of these properties requires Hamiltonian formulations of the models. The Toda theories can be understood as the Hamiltonian reductions of Wess-Zumino-Novikov-Witten (WZNW) models (see ref. [6]). As each WZNW model is integrable, one can explicitly write a Hamiltonian formulation in terms of the coordinates of the space of solutions of the model. In order to preserve full equivalence with the Hamiltonian formulation using the space of initial data, one needs to formulate the WZNW solutions as discussed in ref. [7] (see also ref. [8]). Using this formulation, one may derive, by Hamiltonian reduction, a Hamiltonian formulation of Toda theory [9], utilising suitable coordinates on the space of solutions. It is natural to seek to formulate the supersymmetric analogues of these results.

This would be most easily done in superspace. However, full superspace realisations of generic supersymmetric WZNW models are only available for the $(1, 0)$ and $(1, 1)$ models, and certain $N \geq 2$ models (for $(2, 0)$ and $(2, 1)$ see ref. [10], whilst for $(2, 2)$ and $N > 2$ see ref. [11] for a recent discussion and references). In those cases where a superspace formulation is not available, one can choose to work in some suitable component formulation. By ‘ $N = k$ ’ we will mean (k, k) supersymmetry ($k = 1$ or 2) in the following.

The Hamiltonian formulation of the WZNW models of ref. [7] was recently generalised to the $N = 1$ and some $N = 2$ super WZNW models in ref. [12]. In this paper, we will show how one may apply a Hamiltonian reduction procedure to these super WZNW models to obtain new Hamiltonian formulations of the super Toda theories.

In section 2 we will review the formulation of Hamiltonian reduction as applied to $N = 1$ super WZNW models, and will show how this may be done for the $N = 2$ models (for the $N = 2$ Liouville model, this was done in ref. [13]). Then, in section 3, we will show how this procedure applied to the super WZNW solutions presented in ref. [12] yields new formulations of the solutions of super Toda theories. In section 4, we will use these new phase space coordinate systems to write Hamiltonian formulations of the super Toda theories. We finish with some concluding remarks in section 5.

2 Super Toda Field Theory and Hamiltonian Reduction

2.1 $N = 1$ Hamiltonian Reduction

Here we will follow the Lie super-algebra conventions of ref. [14]. The super-algebra \mathcal{A} is one of the set $sl(n+1|n)$, $Osp(2n \pm 1|2n)$, $Osp(2n|2n)$, $Osp(2n+2|2n)$, $n \geq 1$, or $D(2, 1; \alpha)$, $\alpha \neq 0, -1$. In the Chevalley basis we have Cartan generators H_i , $i = 1, \dots, r = \text{rank}(\mathcal{A})$, and fermionic generators $E_{\pm\alpha_i}$ associated to the simple root system $(\alpha_1, \dots, \alpha_r)$. K_{ij} is the Cartan matrix. $\mathcal{A}_+(\mathcal{A}_-)$ denotes the superalgebra spanned by the generators of \mathcal{A} corresponding to the positive (negative) roots respectively, with respect to the Cartan subalgebra \mathcal{H} . $\mathcal{G}_+, \mathcal{G}_-, \mathcal{G}$ are the corresponding groups associated to the algebras $\mathcal{A}_+, \mathcal{A}_-, \mathcal{A}$.

The $N = 1$ super WZNW model has the equations of motion

$$D_-(D_+GG^{-1}) = 0 \quad (1)$$

for $G \in \mathcal{G}$. Our superspace has coordinates $(x^{\pm\pm}, \theta^{\pm})$ and we use the supercovariant derivatives $D_{\pm} = \partial/\partial\theta^{\pm} + \theta^{\pm}\partial_{\pm\pm}$. We will work in Minkowski space-time, with coordinates (t, x) and signature $(-1, 1)$, and take $x^{\pm\pm} = t \pm x$, $\partial_{\pm\pm} = (1/2)(\partial_t \pm \partial_x)$. We now impose the following constraints

$$\begin{aligned} (D_+GG^{-1})|_- &= \kappa\mu_i E_{-\alpha_i}, \\ (G^{-1}D_-G)|_+ &= \kappa\nu_i E_{\alpha_i}, \end{aligned} \quad (2)$$

for α_i simple, where $|_{\pm}$ means the components along the positive (negative) root systems, μ_i and ν_i are constants and κ is a coupling constant. Then, as discussed in ref. [14], under the constraints (2) the $N = 1$ super WZNW field equations (1) reduce to the $N = 1$ super Toda equations of motion

$$D_+D_- \Phi^i - \kappa^2 \mu_i \nu_i e^{K_{ij}\Phi^j} = 0, \quad (i = 1, \dots, r), \quad (3)$$

(the repeated index i is not summed over in this or later equations of motion). As we will later study the $N = 2$ models in components, it is instructive to see how this works for the $N = 1$ case. The basic fields are a bosonic $g \in \mathcal{G}$ and fermions $\psi_{\pm} \in \text{Lie}\mathcal{G}$, the Lie algebra of \mathcal{G} . These are given by $g = G|_0$, $\psi_+ = (D_+GG^{-1})|_0$ and $\psi_- = (G^{-1}D_-G)|_0$, where $|_0$ means to

evaluate at $\theta^\pm = 0$. The equations of motion of the $N = 1$ super WZNW model are $\partial_{--}(\partial_{++}g g^{-1}) = 0, \partial_{--}\psi_+ = \partial_{++}\psi_- = 0$. The (on-shell) $N = 1$ supersymmetry transformations are

$$\begin{aligned}\delta^+ g &= \epsilon^+ \psi_+ g, & \delta^- g &= \epsilon^- g \psi_- \\ \delta^+ \psi_+ &= \epsilon^+ (\partial_{++}g g^{-1} + \psi_+^2), & \delta^- \psi_- &= \epsilon^- (g^{-1} \partial_{--}g - \psi_-^2) \\ \delta^+ \psi_- &= 0, & \delta^- \psi_+ &= 0.\end{aligned}\tag{4}$$

The Hamiltonian reduction constraints are the fermionic constraints

$$\begin{aligned}\psi_+|_- &= \kappa \mu_i E_{-\alpha_i}, \\ \psi_-|_+ &= \kappa \nu_i E_{\alpha_i},\end{aligned}\tag{5}$$

and their supersymmetric partners

$$\begin{aligned}(\partial_{++}g g^{-1} + \psi_+^2)|_- &= 0, \\ (g^{-1} \partial_{--}g - \psi_-^2)|_+ &= 0.\end{aligned}\tag{6}$$

Under these constraints, the super WZNW equations of motion reduce to the $N = 1$ super Toda equations of motion

$$\begin{aligned}\partial_{++}\partial_{--}\phi^i &= -M_i(K_{ij}\chi_+^j)(K_{ik}\chi_-^k)\exp(K_{il}\phi^l) \\ &\quad -M_i M_j K_{ij}\exp(K_{jk}\phi^k)\exp(K_{il}\phi^l) \\ \partial_{++}\chi_-^i &= -M_i K_{ik}\chi_+^k \exp(K_{ij}\phi^j) \\ \partial_{--}\chi_+^i &= M_i K_{ik}\chi_-^k \exp(K_{ij}\phi^j),\end{aligned}\tag{7}$$

where $M_i = \kappa^2 \mu_i \nu_i$ and χ_\pm^i are the supersymmetric partners of ϕ^i . In terms of the superfield Φ^i of equation (3), these fields are given by $\phi^i = \Phi^i|_0$, $\chi_\pm^i = D_\pm \Phi^i|_0$. The auxiliary field $F^i = D_+ D_- \Phi^i|_0$ is eliminated using its (non-propagating) equation of motion in the usual way.

2.2 $N = 2$ Hamiltonian Reduction

There is currently no off-shell superfield formulation of generic $N = 2$ supersymmetric WZNW models, and we will thus work in components. The $N = 2$ supersymmetry requires the supergroup to be $sl(n+1|n)$, for some positive integer n . It proves convenient to use the left and right invariant one forms on

the group manifold, $L = g^{-1}dg$, $R = dg g^{-1}$, satisfying $dR = R^2$, $dL = -L^2$, or in components

$$R_{[I,J]}^A = -\frac{1}{2}f^A{}_{BC}R_I^B R_J^C, \quad L_{[I,J]}^A = \frac{1}{2}f^A{}_{BC}L_I^B L_J^C, \quad (8)$$

where all the indices run from 1 to $\dim \mathcal{G}$, with A, B, C, \dots , Lie algebra indices and I, J, K, \dots , coordinate indices. The vector fields $R_A = R_A^I \partial_I$, $L_A = L_A^I \partial_I$ ($\partial_I = \partial/\partial X^I$, where X^I are the group manifold coordinates) also satisfy

$$[R^A, R^B] = -f^{AB}{}_C R^C, \quad [L^A, L^B] = f^{AB}{}_C L^C, \quad [R^A, L^B] = 0. \quad (9)$$

Finally we have $R_I^A R_B^I = \delta_B^A$, $R_I^A R_A^J = \delta_I^J$, with similar equations for L .

Under the left and right complex structures on the group manifold, the coordinates naturally split into sets with barred or unbarred indices $I \rightarrow (i, \bar{i})$, $A \rightarrow (a, \bar{a})$, etc. Furthermore we have the properties $f_{abc} = 0 = K_{ab}$ and their conjugates.

The $N = 2$ super WZNW model has the bosonic field $g \in \mathcal{G}$, and fermions $\psi_\pm^a, \bar{\psi}_\pm^{\bar{a}} \in \text{Lie } \mathcal{G}$, with equations of motion

$$\partial_{--}(\partial_{++} g g^{-1}) = 0, \quad \partial_{\pm\pm} \psi_\mp^a = 0 = \partial_{\pm\pm} \bar{\psi}_\mp^{\bar{a}}. \quad (10)$$

The right supersymmetry transformations are

$$\begin{aligned} \delta X^I &= \epsilon^+ R_a^I \psi_+^a + \bar{\epsilon}^+ R_{\bar{a}}^I \bar{\psi}_+^{\bar{a}}, \\ \delta \psi_+^a &= \frac{1}{2} \epsilon^+ f^a{}_{bc} \psi_+^b \psi_+^c + \bar{\epsilon}^+ f^a{}_{b\bar{c}} \psi_+^b \bar{\psi}_+^{\bar{c}} + \bar{\epsilon}^+ R_I^a \partial_{++} X^I, \\ \delta \bar{\psi}_+^{\bar{a}} &= \frac{1}{2} \bar{\epsilon}^+ f^{\bar{a}}{}_{\bar{b}\bar{c}} \bar{\psi}_+^{\bar{b}} \bar{\psi}_+^{\bar{c}} + \epsilon^+ f^{\bar{a}}{}_{b\bar{c}} \psi_+^b \bar{\psi}_+^{\bar{c}} + \epsilon^+ R_I^{\bar{a}} \partial_{++} X^I, \\ \delta \psi_-^a &= \delta \bar{\psi}_-^{\bar{a}} = 0, \end{aligned} \quad (11)$$

with the left supersymmetry transformations being given by the above formulæ with the replacements $\psi_+ \rightarrow \psi_-$, $\partial_{++} \rightarrow \partial_{--}$, $f_{ABC} \rightarrow -f_{ABC}$, $R_I^A \rightarrow L_I^A$.

The natural fermionic Hamiltonian reduction constraints to take are

$$\begin{aligned} \psi_+|_- &= \kappa \mu^a E_-^a, & \psi_-|_+ &= \kappa \nu^a E_+^a, \\ \bar{\psi}_+|_- &= \kappa \mu^{\bar{a}} E_-^{\bar{a}}, & \bar{\psi}_-|_+ &= \kappa \nu^{\bar{a}} E_+^{\bar{a}}, \end{aligned} \quad (12)$$

where $|_{\pm}$ means the components along the positive or negative root systems. $E_{\pm}^a, E_{\pm}^{\bar{a}}$ are the fermionic step operators corresponding to the simple roots. We also impose the supersymmetry partners of these constraints, which are

$$\begin{aligned} \psi_+^2|_- &= 0, & \psi_-^2|_+ &= 0, \\ \bar{\psi}_+^2|_- &= 0, & \bar{\psi}_-^2|_+ &= 0, \\ (\partial_{++}g g^{-1} + \{\psi_+, \bar{\psi}_+\})|_- &= 0, \\ (g^{-1}\partial_{--}g - \{\psi_-, \bar{\psi}_-\})|_+ &= 0. \end{aligned} \tag{13}$$

Under the constraints (12) and (13), the WZNW system (10) reduces to the $N = 2$ super Toda theory whose equations of motion will be given below in Section 3.2.

We have not discussed possible Hamiltonian reductions of ‘chiral’ WZNW models, with (p, q) , $p \neq q$, supersymmetry. Lagrangian formulations of these models exist, as noted in the introduction, and one may impose constraints to reduce these theories to systems with Toda-type equations of motion. However, to our knowledge there are no Lagrangian formulations of (p, q) , $p \neq q$ supersymmetric Toda theories and for this reason we have not further investigated these chiral reductions.

3 The Super Toda Solutions

In the previous section it was shown how to impose constraints in order to reduce the $N = 1$ and 2 super WZNW models to the super Toda theories. One may use this to reduce solutions of super WZNW models to solutions of super Toda theories. The former have been presented in ref. [12]. In this section, we will show how to obtain new formulations of the solutions of $N = 1, 2$ super Toda theories by Hamiltonian reduction of these WZNW solutions.

3.1 $N = 1$ Super Toda Solutions

For the $N = 1$ model we begin with the solution of ref. [7] to the field equations (1), *i.e.*

$$G(t, x, \theta^+) = U(x^{++}, \theta^+) W(A; x^{++}, x^{--}) V(x^{--}, \theta^-),$$

$$W(A; x^{++}, x^{--}) = P \exp \int_{x^{--}}^{x^{++}} A(s) ds, \quad (14)$$

with A a $(\text{Lie}\mathcal{G})^*$ -valued one-form on the real line, P denoting path ordering. The parameterisation (14) of the solutions is invariant under the group transformations

$$\begin{aligned} U(x, \theta^+) &\longrightarrow U(x, \theta^+)h(x), \quad V(x, \theta^-) \longrightarrow h^{-1}(x)V(x, \theta^-), \\ A(x) &\longrightarrow -h^{-1}(x)\partial_x h(x) + h^{-1}(x)A(x)h(x), \end{aligned} \quad (15)$$

where h is an element of the loop group of \mathcal{G} .

Rewriting the constraints (2) in terms of the variables U, V, A of the parameterisation (14), they become

$$\begin{aligned} (D_+ U U^{-1} + U A(x^{++}) U^{-1})|_- &= \kappa \mu_i E_{-\alpha_i}, \\ (V^{-1} D_- V - V^{-1} A(x^{--}) V)|_+ &= \kappa \nu_i E_{\alpha_i}. \end{aligned} \quad (16)$$

To solve these constraints, we recall that the group \mathcal{G} admits (locally) a Gauss decomposition, so that elements $G \in \mathcal{G}$ can be decomposed as $G = DBC$, where D and C lie in \mathcal{G}_+ and \mathcal{G}_- respectively, and B lies in a maximal torus of \mathcal{G} with Lie algebra \mathcal{H} . The parameters U, V can thus be decomposed as

$$U = D_L B_L C_L, \quad V = D_R B_R C_R, \quad (17)$$

for some

$$\begin{aligned} D_L &= \exp\left(\sum_{\alpha \in \Pi_+} \hat{X}_L^\alpha E_\alpha\right), \quad C_L = \exp\left(\sum_{\alpha \in \Pi_+} \hat{Y}_L^\alpha E_{-\alpha}\right), \\ B_L &= \exp(\hat{\Phi}_L^i H_i), \end{aligned} \quad (18)$$

(Π_+ is the set of positive roots) and similarly for V , with left subscripts L replaced by right subscripts R . The constraints (16) can be rewritten in terms of $D_R, \hat{\Phi}_L, \hat{\Phi}_R$ and C_L as follows:

$$\begin{aligned} (D_+ C_L C_L^{-1} + \theta^+ C_L A(x^{++}) C_L^{-1})|_- &= \sum_{\alpha_i \in \Delta_+} \mu_i E_{-\alpha_i} \exp(K_{ij} \hat{\Phi}_L^j), \\ (D_R^{-1} D_- D_R - \theta^- D_R^{-1} A(x^{--}) D_R)|_+ &= - \sum_{\alpha_i \in \Delta_+} \nu_i E_{\alpha_i} \exp(K_{ij} \hat{\Phi}_R^j), \end{aligned} \quad (19)$$

with Δ_+ the set of simple roots.

To find the independent parameters of the solutions of the Toda equations, we have still to gauge-fix the symmetry (15) and solve the constraints (19). The transformation (15) can be gauge-fixed by setting the θ^+ independent parts of U to one. Next, as the constraints (19) do not contain C_R and D_L , we can set $C_R = D_L = 1$ (alternatively this can be understood as fixing the associated shift symmetry). The gauge symmetry

$$\begin{aligned} D_R(x^{--}, \theta^-) &\longrightarrow h_+^{-1}(x^{--})D_R(x^{--}, \theta^-), \\ A(s) &\longrightarrow -h_+^{-1}(s)\partial_s h_+(s) + h_+^{-1}(s)A(s)h_+(s), \end{aligned} \quad (20)$$

where h_+ lies in the loop group of \mathcal{G}_+ , leaves G and the constraints (19) invariant. We gauge fix this symmetry by setting the θ^- independent component of D_R to 1. We use a similar gauge symmetry for C_L to set the θ^+ independent component of C_L to 1. Hence, after gauge-fixing we can write

$$\begin{aligned} D_L &= 1, \quad B_L = 1 + \theta^+ \psi_+^i(x^{++})H_i, \\ C_L &= 1 + \theta^+ \sum_{\alpha \in \Pi_+} Y_L^{\alpha_i}(x^{++})E_{-\alpha_i}, \quad D_R = 1 + \theta^- \sum_{\alpha \in \Pi_+} X_R^{\alpha_i}(x^{--})E_{\alpha_i}, \\ B_R &= \exp(\hat{\Phi}_R^i(x^{--}, \theta^-)H_i), \quad C_R = 1, \end{aligned} \quad (21)$$

for some (super-)fields $\psi_+^i(x^{++})$, $Y_L^\alpha(x^{++})$, $X_R^\alpha(x^{--})$, $\hat{\Phi}_R^i(x^{--}, \theta^-)$. Solving for these fields by substituting the expressions (21) into the constraints (19) then leads straightforwardly to the final gauge fixed parameterisation

$$\begin{aligned} U &= 1 + \theta^+ (\psi_+^i H_i + \kappa \mu_i E_{-\alpha_i}), \\ V &= [1 + \theta^- (\psi_-^i H_i + \kappa \nu_i \exp(K_{ij} \phi_R^j) E_{\alpha_i})] \exp(\phi_R^k H_k), \\ A &= \hat{A} \equiv a^i H_i - \kappa \mu_i (K_{ij} \psi_+^j) E_{-\alpha_i} + \kappa \nu_i (K_{ik} \psi_-^k) \exp(K_{ij} \phi_R^j) E_{\alpha_i} \\ &\quad - (\kappa \mu_i E_{-\alpha_i})^2 - (\kappa \nu_i \exp(K_{ij} \phi_R^j) E_{\alpha_i})^2, \end{aligned} \quad (22)$$

where we have expanded $\hat{\Phi}_R^i(x^{--}, \theta^-) = \phi_R^i(x^{--}) + \theta^- \psi_-^i(x^{--})$.

The field

$$G = U \exp\left(\int_{x^-}^{x^+} A(s) ds\right) V, \quad (23)$$

with the expressions (22) inserted, is then the solution of the system of equations (1) and (2). Using this we can directly obtain the $N = 1$ super Toda solution as follows. We first perform a Gauss decomposition for the WZNW field G

$$G = DBC, \quad (24)$$

where

$$\begin{aligned} D &= \exp\left(\sum_{\alpha \in \Pi_+} x^\alpha E_\alpha\right), \quad C = \exp\left(\sum_{\alpha \in \Pi_+} y^\alpha E_{-\alpha}\right), \\ B &= \exp(\phi^i H_i), \end{aligned} \quad (25)$$

for some fields $x^\alpha, y^\alpha, \phi^i$. The Toda field is identified as the field ϕ that appears in the definition of B in (25). To project it from the expression for G in (24) we use the standard method of the l normalised lowest weight states $|\lambda_i\rangle$ of finite dimensional representations of G (with $H_j|\lambda_i\rangle = -\delta_{ij}|\lambda_i\rangle$). From equations (25), (14), and $\langle l_i|G|l_i\rangle = \exp(-\Phi^i)$ we obtain the result

$$\begin{aligned} \exp(-\Phi^i(\theta^+, \theta^-, x, t)) &= \exp(-\phi_R^i \langle \lambda_i | [1 + \theta^+(\psi_+^j H_j + \kappa \mu_j E_{-\alpha_j})] \times \\ &\quad \mathcal{W}(\hat{A}; x^{++}, x^{--}) \times [1 + \theta^-(\kappa \nu_j \exp(K_{jk} \phi_R^k) E_{\alpha_j} + \psi_-^j H_j)] | \lambda_i \rangle). \end{aligned} \quad (26)$$

The free parameters of this formulation of the super Toda solutions are the chiral fields $\psi_+^i(x^{++}), \psi_-^i(x^{--}), \phi_R^i(x^{--})$ and the field $a^i(s)$ which appears in $\hat{A}(s)$ in $W(\hat{A}, x^{++}, x^{--})$.

The solutions for the component fields of Φ are readily found from equation (26) and are given by

$$\begin{aligned} \phi^i &\equiv \Phi^i|_0 = \phi_R^i - \log \langle \lambda_i | W(\hat{A}) | \lambda_i \rangle, \\ \chi_+^i &\equiv D_+ \Phi|_0 = \frac{\langle \lambda_i | (\psi_+ \cdot H + \kappa \mu \cdot E_-) W(\hat{A}) | \lambda_i \rangle}{\langle \lambda_i | W(\hat{A}) | \lambda_i \rangle}, \\ \chi_-^i &\equiv D_- \Phi|_0 = \frac{\langle \lambda_i | W(\hat{A}) (\psi_- \cdot H + \kappa \exp(K_{ij} \phi_R^j) \nu \cdot E_+) | \lambda_i \rangle}{\langle \lambda_i | W(\hat{A}) | \lambda_i \rangle}. \end{aligned} \quad (27)$$

By construction, these fields satisfy the equations of motion (7). A direct proof of this also follows by utilising the Lie algebraic techniques described in Section 2 of ref. [15].

3.2 The $N = 2$ Super Toda Solutions

To begin with, we note that the $N = 2$ WZNW equations of motion (10) are solved by

$$g(x, t) = u(x^{++})W(A)v(x^{--}), \quad W(A) = \exp\left(\int_{x^{--}}^{x^{++}} A(s) ds\right),$$

$$\begin{aligned}
\psi_+^a(x, t) &= \psi_L^a(x^{++}), & \bar{\psi}_+^{\bar{a}}(x, t) &= \bar{\psi}_L^{\bar{a}}(x^{++}), \\
\psi_-^a(x, t) &= \psi_R^a(x^{--}), & \bar{\psi}_-^{\bar{a}}(x, t) &= \bar{\psi}_R^{\bar{a}}(x^{--}).
\end{aligned} \tag{28}$$

Inserting these into the constraints (12-13), gives

$$\begin{aligned}
\psi_L|_- &= \kappa\mu^a E_-^a, & \psi_R|_+ &= \kappa\nu^a E_+^a, \\
\bar{\psi}_L|_- &= \kappa\mu^{\bar{a}} E_-^{\bar{a}}, & \bar{\psi}_R|_+ &= \kappa\nu^{\bar{a}} E_+^{\bar{a}}, \\
\psi_L^2|_- &= 0 = \bar{\psi}_L^2|_-, \\
\psi_R^2|_+ &= 0 = \bar{\psi}_R^2|_+, \\
(\partial_{++}u u^{-1} + uA(x^{++})u^{-1} + \{\psi_L, \bar{\psi}_L\})|_- &= 0, \\
(v^{-1}\partial_{--}v - v^{-1}A(x^{--})v - \{\psi_R, \bar{\psi}_R\})|_- &= 0.
\end{aligned} \tag{29}$$

Now we perform Gauss decompositions as in the $N = 1$ supersymmetric case. Writing $u = d_L b_L c_L$ and $v = d_R b_R c_R$, the bosonic constraints may be shown to reduce to

$$\begin{aligned}
(\partial_{++}c_L c_L^{-1} + c_L A(x^{++})c_L^{-1})|_- &= -(b_L^{-1}d_L^{-1}\{\psi_L, \bar{\psi}_L\}d_L b_L)|_-, \\
(d_R^{-1}\partial_{--}d_R - d_R^{-1}A(x^{--})d_R)|_- &= (b_R c_R \{\psi_R, \bar{\psi}_R\}c_R^{-1}b_R^{-1})|_-.
\end{aligned} \tag{30}$$

We now fix the gauge symmetries of the parameterisations and constraints, in a similar way to the $N = 1$ case. With regard to the fermionic constraints in equation (29), these are easily solved. Then we fix $u = 1$ (so that $c_L = b_L = d_L = 1$) by the gauge symmetry in the solution (28). Next we set to zero the fields $\bar{\psi}_L^{\bar{a}}, \psi_L^a$ ($\bar{\psi}_R^{\bar{a}}, \psi_R^a$), for $\alpha_{\bar{a},a}$ the positive (negative of the) simple roots, as these fields do not appear in the constraints. We similarly set the components of $\psi_L, \bar{\psi}_L$ along the directions of the negative non-simple roots to zero, and the components of $\psi_R, \bar{\psi}_R$ along the directions of the positive non-simple roots to zero. The second equation in equation (30) above has a triangular symmetry whereby one may set $d_R = 1$. Finally, on the right-hand side of the second equation in equation (30) above, if one considers the coefficients of the fermionic generators corresponding to the positive roots, one finds a shift symmetry which can be used to gauge away the only piece of c_R which contributes to this expression. Thus one may set $c_R = 1$ in this equation.

Thus we come to the following solution of the constraints (29)

$$\psi_L = \kappa\mu^a E_-^a + \xi_L^i H^i,$$

$$\begin{aligned}
\bar{\psi}_L &= \kappa\mu^{\bar{a}}E_{-}^{\bar{a}} + \xi_L^{\bar{i}}H^{\bar{i}}, \\
\psi_R &= \kappa\nu^aE_{+}^a + \xi_R^iH^i, \\
\bar{\psi}_R &= \kappa\nu^{\bar{a}}E_{+}^{\bar{a}} + \xi_R^{\bar{i}}H^{\bar{i}}, \\
A|_H &= a^iH^i + a^{\bar{i}}H^{\bar{i}}, \\
A|_{a,\bar{a}<0} &= -\{\psi_L, \bar{\psi}_L\}|_{a,\bar{a}<0}, \\
A|_{a,\bar{a}>0} &= (b_R\{\psi_R, \bar{\psi}_R\}b_R^{-1})|_{a,\bar{a}>0},
\end{aligned} \tag{31}$$

with the other components of A equal to zero. ‘ $|_{a,\bar{a}>0}$ ’ means the components along the directions of $E_{+}^{\bar{a}}, E_{+}^a$, and similarly ‘ $|_{a,\bar{a}<0}$ ’ means the components along the directions of $E_{-}^{\bar{a}}, E_{-}^a$. The “coordinates” of this parameterisation of the constraint solution are the fields $\xi_R^i(x^{--}), \xi_R^{\bar{i}}(x^{--}), \xi_L^i(x^{++}), \xi_L^{\bar{i}}(x^{++}), a^i(s), a^{\bar{i}}(s), \phi_R^i(x^{--})$ and $\phi_R^{\bar{i}}(x^{--})$. We may reconstruct the expression for the field g using $g = W(\hat{A})\exp(\phi_R \cdot H)$, where the field \hat{A} in $W(\hat{A})$ has non-zero components given by the expressions for the components of A in equation (31).

The explicit expressions for the Toda fields in terms of the solution parameters are then

$$\begin{aligned}
\phi^i(x, t) &= \phi_R^i(x^{--}) - \log\langle\lambda_i|W(\hat{A})|\lambda_i\rangle, \\
\bar{\phi}^{\bar{i}}(x, t) &= \phi_R^{\bar{i}}(x^{--}) - \log\langle\lambda_{\bar{i}}|W(\hat{A})|\lambda_{\bar{i}}\rangle, \\
\chi_+^i(x, t) &= \frac{\langle\lambda_i|(\kappa\mu^aE_{-}^a + \xi_L^jH^j)W(\hat{A})|\lambda_i\rangle}{\langle\lambda_i|W(\hat{A})|\lambda_i\rangle}, \\
\chi_+^{\bar{i}}(x, t) &= \frac{\langle\lambda_{\bar{i}}|(\kappa\mu^{\bar{a}}E_{-}^{\bar{a}} + \xi_L^{\bar{j}}H^{\bar{j}})W(\hat{A})|\lambda_{\bar{i}}\rangle}{\langle\lambda_{\bar{i}}|W(\hat{A})|\lambda_{\bar{i}}\rangle}, \\
\chi_-^i(x, t) &= \frac{\langle\lambda_i|W(\hat{A})(\kappa\exp(K_{a\bar{b}}\bar{\phi}_R^{\bar{b}})\nu^aE_{+}^a + \xi_R^iH^i)|\lambda_i\rangle}{\langle\lambda_i|W(\hat{A})|\lambda_i\rangle}, \\
\chi_-^{\bar{i}}(x, t) &= \frac{\langle\lambda_{\bar{i}}|W(\hat{A})(\kappa\exp(K_{\bar{a}b}\phi_R^b)\nu^{\bar{a}}E_{+}^{\bar{a}} + \xi_R^{\bar{i}}H^{\bar{i}})|\lambda_{\bar{i}}\rangle}{\langle\lambda_{\bar{i}}|W(\hat{A})|\lambda_{\bar{i}}\rangle}.
\end{aligned} \tag{32}$$

These fields satisfy the $N = 2$ Toda field equations

$$\begin{aligned}
\partial_{++}\partial_{--}\phi^i &= \alpha_i(K_{i\bar{j}}\bar{\chi}_{-}^{\bar{j}})(K_{i\bar{k}}\bar{\chi}_{+}^{\bar{k}})\exp(K_{i\bar{l}}\bar{\phi}^{\bar{l}}) \\
&\quad - \alpha_i\alpha_{\bar{j}}K_{i\bar{j}}\exp(K_{\bar{j}l}\phi^l + K_{i\bar{k}}\bar{\phi}^{\bar{k}}), \\
\partial_{--}\chi_+^i &= \alpha_i(K_{i\bar{j}}\bar{\chi}_{-}^{\bar{j}})\exp(K_{i\bar{k}}\bar{\phi}^{\bar{k}}), \\
\partial_{++}\chi_-^i &= -\alpha_i(K_{i\bar{j}}\bar{\chi}_{+}^{\bar{j}})\exp(K_{i\bar{k}}\bar{\phi}^{\bar{k}}),
\end{aligned} \tag{33}$$

and their conjugates (which arise from interchanging barred and unbarred quantities). In the above, $\alpha_i = -\kappa^2 \mu^i \nu^i$. We may write equation (33) in $N = 2$ superspace as follows. Define $D_+ = \frac{\partial}{\partial \theta^+} + \bar{\theta}^+ \partial_{++}$ and $D_- = \frac{\partial}{\partial \theta^-} + \bar{\theta}^- \partial_{--}$, and their conjugates. Introduce superfields Φ^i and $\bar{\Phi}^{\bar{i}}$ satisfying the chirality constraints $D_\pm \bar{\Phi}^{\bar{i}} = 0$ and $\bar{D}_\pm \Phi^i = 0$. Then make the component-field definitions $\phi^i = \Phi^i|_0$, $\chi_\pm^i = D_\pm \Phi^i|_0$ and the conjugates of these equations. Then the equations of motion (33) may be written in $N = 2$ superspace as

$$D_+ D_- \Phi^i = \alpha_i \exp(K_{ij} \bar{\Phi}^{\bar{j}}), \quad (34)$$

(and its conjugate) where the auxiliary fields are eliminated using their non-propagating equations of motion.

4 The Phase Spaces of Super Toda Theory

Well-defined Hamiltonian formulations of the super WZNW models will reduce under (first class) constraints to well-defined Hamiltonian formulations of the super Toda theories. Utilising the space of solutions as the phase space we may thus use the super Toda solutions derived in the previous section to parameterise the super Toda phase spaces. It remains to give the Poisson brackets for the variables defining the solutions, which we will now do.

The symplectic form for the $N = 1$ super Toda theory is

$$\Omega = \frac{1}{2} \int dx \left(\int d\theta^+ (K_{ij} \delta \Phi^i D_+ \delta \Phi^j)|_{\theta^- = 0} - \int d\theta^- (K_{ij} \delta \Phi^i D_- \delta \Phi^j)|_{\theta^+ = 0} \right). \quad (35)$$

One can check that this is closed and time independent, using the equations of motion (3). This is most directly done in components. Now we insert the solution (26) (the complexity of this calculation is much reduced by using the time independence of the symplectic form to set $t = 0$ in the Toda solutions and their derivatives). This leads to

$$\Omega = \int dx K_{ij} (\delta \phi_R^i \delta (\phi_R^j + 2a^j) + \frac{1}{2} \delta \psi_+^i \delta \psi_+^j - \frac{1}{2} \delta \psi_-^i \delta \psi_-^j). \quad (36)$$

Thus the Poisson brackets become those of the non-supersymmetric Toda model, with parameters (ϕ_R^i, a^i) (*c.f.* ref. [9]), together with free fermion Poisson brackets for the fields ψ_\pm^i .

The $N = 2$ case is similar to the $N = 1$ case just presented. The symplectic form of the $N = 2$ Toda theory is

$$\Omega = \int dx K_{i\bar{j}} \left(\delta\phi^i \delta\dot{\bar{\phi}}^{\bar{j}} + \delta\bar{\phi}^{\bar{j}} \delta\dot{\phi}^i + \delta\chi_+^i \delta\chi_+^{\bar{j}} - \delta\chi_-^i \delta\chi_-^{\bar{j}} \right). \quad (37)$$

One can check that this is closed and time independent, using the Toda equations of motion. It is also straightforward to check that it is $N = 2$ supersymmetric - for example, one of the right supersymmetries is given by the transformations $\Delta\phi^i = \epsilon^+ \chi_+^i$, $\Delta\chi_+^{\bar{i}} = \epsilon^+ \partial_{++} \bar{\phi}^{\bar{i}}$, with the other variations zero, and one can show directly that $\Delta\Omega = 0$. Inserting the Toda solutions (32) into the symplectic form (37) gives in a straightforward way

$$\Omega = \int dx K_{i\bar{j}} \left(\delta\phi_R^i \delta(\bar{\phi}_R^{\bar{j}} + 2a^{\bar{j}}) + \delta\bar{\phi}_R^{\bar{j}} \delta(\phi_R^i + 2a^i) + \delta\xi_+^i \delta\xi_+^{\bar{j}} - \delta\xi_-^i \delta\xi_-^{\bar{j}} \right). \quad (38)$$

Thus we see again the Poisson brackets of the bosonic model together with free field brackets for the fermions. In the $N = 1$ and $N = 2$ cases we see from equations (36) and (38) that the solution parameter variables are simply related to the usual canonical Hamiltonian variables (the initial data).

5 Concluding Remarks

In this paper we have presented new formulations of the solutions of the $N = 1$ and $N = 2$ supersymmetric Toda theories. These were obtained using Hamiltonian reduction on formulations of the solutions of the corresponding WZNW models which we presented in a previous paper. The advantage of our formulations of the super Toda solutions is that they allow a direct correspondence between the solutions and the initial data. Using this, one can then formulate the phase space description using the space of solutions, which can then be seen to be equivalent to the Hamiltonian formulation.

Although we have not presented details here, these methods can also be applied directly to other Toda systems - the various models obtained by non-standard reductions, and the models obtained by reductions of the affine WZNW theories, for example.

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